Sinh and Cosh Fact Sheet

1 Definition

We define

$$\sinh t = \frac{e^t - e^{-t}}{2} \quad \text{and} \quad \cosh t = \frac{e^t + e^{-t}}{2}.$$

2 Similarities to Sine and Cosine

Note that

$$\cosh^2 t - \sinh^2 t = \frac{e^{2t} + 1 + e^{-2t}}{2} - \frac{e^{2t} - 1 + e^{-2t}}{2} = 1.$$

That is, for any $t \in \mathbb{R}$, $(\cosh t, \sinh t)$ is a point on the unit hyperbola $x^2 - y^2 = 1$, just as $(\cos t, \sin t)$ is a point on the unit circle $x^2 + y^2 = 1$.

3 Parity

Note that $\cosh(-t) = \cosh t$ and $\sinh(-t) = -\sinh t$, so $\cosh t$ is an even function and $\sinh t$ is an odd function, just as $\cos t$ is even while $\sin t$ is odd.

4 Derivatives

Using the basic rules for differentiation, we find

$$D_t\left[\sinh t\right] = D_t\left[\frac{e^t - e^{-t}}{2}\right] = \frac{e^t + e^{-t}}{2} = \cosh t \quad \text{and}$$
$$D_t\left[\cosh t\right] = D_t\left[\frac{e^t + e^{-t}}{2}\right] = \frac{e^t - e^{-t}}{2} = \sinh t.$$

Thus $\cosh t$ and $\sinh t$ are both solutions to the ODE $u'' - u = 0$. Note the similarity with $\cos t$ and $\sin t$ which both solve the ODE $u'' + u = 0$.

5 Maclaurin Series

Recall

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots,$$

and so

$$e^{-t} = 1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - + \cdots.$$

Thus

$$\cosh t = \frac{e^t + e^{-t}}{2} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \quad \text{and}$$
$$\sinh t = \frac{e^t - e^{-t}}{2} = \frac{t}{1!} + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots.$$

Compare this with the expansions for $\cos t$ and $\sin t$:

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - + \cdots \quad \text{and}$$
$$\sin t = \frac{t}{1!} - \frac{t^3}{3!} + \frac{t^5}{5!} - + \cdots.$$

6 Euler's Formula

It follows from the Maclaurin expansions above that $\cosh it = \cos t$ and $\sinh it = i \sin t$. Since

$$\cosh it + \sinh it = \frac{e^{it} + e^{-it}}{2} + \frac{e^{it} - e^{-it}}{2} = e^{it},$$

it follows that

$$e^{it} = \cos t + i \sin t.$$

This result is called Euler's Formula, and using it we can define exponentiation and trigonometric functions for all complex numbers. Euler's Identity is an immediate and beautiful corollary:

$$e^{i\pi} + 1 = 0.$$