Review

for 263.02 final

14.1 Functions of several variables.

- Find domain and range. Evaluate.
- Sketch a graph. Draw and interpret level curves. (Functions of three variables have level surfaces.)
- Match surfaces with level curves.

14.2 Limits and continuity.

- The limit is undefined if two paths to a point suggest different values.
- Compute limits (various techniques: multiply and divide by conjugate, squeeze theorem, convert to polar, etc. . .).
- A function f is continuous at a point if the limit and the function value both exist and are equal. Find points of continuity.

14.3 Partial derivatives.

- Find partial derivatives by differentiating with respect to one variable while treating the others as constants.
- Estimate derivatives from graph or contours.
- Mixed partials. Clairaut's theorem: If f_{xy} and f_{yx} are both continuous in D, then $f_{xy} = f_{yx}$ in D.
- Implicit differentiation.
- Check solutions for partial differential equations by substitution.

14.4 Tangent planes and linear approximations.

- Find tangent plane at a point.
- The total differential: $\mathrm{d}z = \dfrac{\partial z}{\partial x}\,\mathrm{d}x + \dfrac{\partial z}{\partial y}\,\mathrm{d}y$, and similarly in higher dimensions.
- Use the total differential to estimate errors, approximate functions.

14.5 The chain rule.

- For $z = f(x, y)$, $x = g(t)$, $y = h(t)$, we have $\dfrac{\mathrm{d}z}{\mathrm{d}t} = \dfrac{\partial z}{\partial x}\dfrac{\mathrm{d}x}{\mathrm{d}t} + \dfrac{\partial z}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}t}$.
- For $z = f(x, y)$, $x = g(s, t)$, $y = h(s, t)$, we have $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$. We find $\partial z/\partial s$ similarly.
- Use chain rule for higher order derivatives as well.

14.6 Directional derivatives and the gradient vector.

- The gradient of f: $\nabla f(x, y) = \langle f_x, f_y\rangle$. A similar statement applies in 3+ dimensions. Gradient vector points in path of fastest increase, so moving in the direction of the gradient gives the path of steepest ascent.
- Directional derivative: $D_{\mathbf{u}} = \nabla f \cdot \mathbf{u}$, where $\mathbf{u}$ is a unit vector.
- Find tangent planes to level surfaces $F(x, y, z) = k$. Gradient of F is in normal direction to the surface.

14.7 Maximum and minimum values.

- f_x, f_y are 0 at a local max or min (provided they exist). Critical points are the places where all first order derivatives are 0.
- Second derivatives test. Consider $D = f_{xx}\,f_{yy} - f_{xy}^2$. If $D > 0, f_{xx} > 0$ at a critical point, then local min. If $D > 0, f_{xx} < 0$ at a critical point, then local max. If $D < 0$, neither a local max nor min.
- To find absolute max or min, check critical points and the boundary.

14.8 Lagrange multipliers.

- Know method. Used for maximizing and minimizing subject to one or more constraints, e.g. $g(x, y, z) = k$.

15.1 Double integrals over rectangles.

- Double Riemann sum definition.
- Approximate via midpoint method.
- Average value: $f_{\text{average}} = \dfrac{1}{\text{Area}(R)} \iint\limits_R f(x, y)\,\mathrm{d}A$.
- Basic properties: linearity, order preserving.

15.2 Iterated integrals.

- Fubini's theorem:
  If $R = [a, b] \times [c, d]$, then $\iint\limits_R f(x, y)\,\mathrm{d}A = \int_a^b \int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y$,
  i.e. $\mathrm{d}A$ becomes $\mathrm{d}x\,\mathrm{d}y$.
- Evaluate inside integral by treating other variables as constants. If $f(x, y) = g(x)\,h(y)$, can evaluate as the product of two integrals.

15.3 Double integrals over general regions.

- Extend functions over a general region to be over a rectangle by taking them to be 0 outside of their domain.
- Break a region into pieces to make easier to integrate. Bounds for inner integrals may depend on outer variables, but not the other way.
- Exchange the order of integration when necessary.
- Properties of double integrals.
- Integrating 1 gives area.

15.4 Double integrals in polar coordinates.

- For f defined in the polar region $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, $\iint\limits_R f\,\mathrm{d}A = \int_\alpha^\beta \int_a^b f\,r\,\mathrm{d}r\,\mathrm{d}\theta$, i.e. $\mathrm{d}A$ becomes $r\,\mathrm{d}r\,\mathrm{d}\theta$.
- Use $x = r\,\cos(\theta)$ and $y = r\,\sin(\theta)$ to convert cartesian problems to polar.

15.5 Applications of double integrals.

- Mass is the integral of density.
- For density $\rho(x, y)$, moment about the x-axis is $M_x = \iint\limits_D y\,\rho(x, y)\,\mathrm{d}A$. Similarly, $M_y = \iint\limits_D x\,\rho(x, y)\,\mathrm{d}A$.

- Center of mass is at $(M_y/m, M_x/m)$. (Yes, M_y goes with the x-coordinate, and M_x with the y.)
- For moment of inertia (second moment), $I_x = \iint\limits_D y^2\,\rho(x,y)\,\mathrm{d}A$. Similarly for I_y. Moment of inertia about the origin is $I_0 = I_x + I_y$.
- Probability within a region is the integral of the joint density function.
- Compute expected values given a joint density function.

15.6 Triple integrals.

- Fubini's theorem extends to higher dimensions. (When integrating over a box, integrate the x, the y, and the z.)
- Define integral in a general region by working in a box, taking the function to be zero inside the box but outside the old domain.
- $\mathrm{d}V$ becomes $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.
- Integrating 1 gives volume.
- Iterated integrals as with two variables.
- Compute probabilities.

15.7 Triple integrals in cylindrical coordinates.

- Especially useful for solids of revolution.
- $\mathrm{d}V$ becomes $r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$.

15.8 Triple integrals in spherical coordinates.

- Especially useful for cones and spheres centered at the origin.
- $\mathrm{d}V$ becomes $\rho^2\,\sin(\phi)\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$.
- $r = \rho\,\sin(\phi)$, so $x = \rho\,\sin(\phi)\,\cos(\theta)$, $y = \rho\,\sin(\phi)\,\sin(\theta)$, and $z = \rho\,\cos(\phi)$. Hence $x^2+y^2+z^2 = \rho^2$.
- Note that $0 \leqslant \phi \leqslant \pi$ and $0 \leqslant \theta < 2\,\pi$.

15.9 Change of variables in multiple integrals.

- The Jacobian of the transformation $x = g(u,v)$ and $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix},$$

  with a similar definition holding for transformations of three or more variables. (Note: The Jacobian is a scalar valued function.)
- $\iint\limits_R f(x,y)\,\mathrm{d}A = \iint\limits_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$, i.e. $\mathrm{d}A = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$. A similar statement holds for triple integrals.

16.1 Vector fields.

- Sketch vector fields: A vector field $\mathbf{F}$ is a function that assigns a vector to every point in its domain. The output is the same dimension as the input.

16.2 Line integrals.

- If C is the parametrically defined curve $x = x(t)$, $y = y(t)$, $a \leqslant t \leqslant b$, then

$$\int\limits_C f(x,y)\,\mathrm{d}s = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

- Instead of integrating with respect to arc length s, we can integrate with respect to x :

$$\int_C f(x,y)\,\mathrm{d}x = \int_a^b f(x(t),y(t))\,x'(t)\,\mathrm{d}t,$$

and similarly for integrating with respect to y. (Follows from the chain rule.)
- Integrating 1 with respect to arc length gives the total arc length.
- Line integral of vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ along C is

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t = \int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

- Work to move a particle along the curve C defined by $\mathbf{r}(t)$ is $W = \int_a^b \mathbf{F}\cdot\mathrm{d}\mathbf{r}$.

16.3 The fundamental theorem for line integrals.

- A conservative vector field is a field $\mathbf{F} = \nabla f$ for some function f.
- $\int_C \nabla f\cdot\mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. That is, the line integral of a conservative vector field is independent of path. Conversely, if a line integral of a continuous vector field $\mathbf{F}$ is independent of path, then $\mathbf{F}$ is a conservative vector field.
- $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ is independent of path if and only if $\int_L \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 0$ for every closed path L in the domain.
- If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is a conservative vector field and P and Q have continuous derivatives, then $P_y = Q_x$. (Statement is true on any domain; converse only holds for open simply-connected sets.)
- If force is described by a conservative vector field, then energy is preserved. (Conservation of energy.)

16.4 Green's theorem.

- Converts line integrals over the boundary to integrals over the area.
- $\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D (Q_x - P_y)\,\mathrm{d}A = \iint_D \operatorname{curl}(P\,\mathbf{i} + Q\,\mathbf{j})\cdot\mathbf{k}\,\mathrm{d}A$, for C positively oriented.
  - If C is negatively oriented, then sign is flipped from the above.
  - Positive orientation: Counterclockwise rotation. (Region to left of direction of motion.)
  - Negative orientation: Clockwise rotation. (Region to right of direction of motion.)
- Sometimes useful to calculate areas enclosed by parametric curves. Just pick any Q and P such that $Q_x - P_y = 1$. Examples include: $Q = x$ and $P = 0$, $Q = 0$ and $P = -y$, or $Q = x/2$ and $P = -y/2$.

16.5 Curl and divergence.

- Calculate curl and divergence: $\operatorname{curl}(\mathbf{F}) = \nabla\times\mathbf{F}$. $\operatorname{div}(\mathbf{F}) = \nabla\cdot\mathbf{F}$.
- Positive divergence at P means net flow near P is outward.
- $(\operatorname{curl}(\mathbf{v}))(P)$ points in the direction of the axis of rotation of $\mathbf{v}$ at P.
- $\operatorname{curl}(\nabla f) = \mathbf{0}$, provided f has continuous second derivatives. That is, the curl of a conservative vector field is $\mathbf{0}$. The converse is true as well: If $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.
- $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$.

- Vector forms of Green's theorem:

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_D (\mathrm{curl}\,(\mathbf{F})\cdot \mathbf{k})\ \mathrm{d}A, \quad \oint_C (\mathbf{F}\cdot \mathbf{n})\ \mathrm{d}s = \iint_D \mathrm{div}\,(\mathbf{F})\ \mathrm{d}A.$$

16.6 Parametric surfaces and their areas.

- Find parametric representation for surfaces.
- Find tangent plane to surface. $\mathbf{r}_u \times \mathbf{r}_v$ is the normal vector.
- Area of surface defined by $\mathbf{r}(u,v)$ where $(u,v)\in D$ is $A = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}A$.
- Special case: Area of surface $z = f(x,y)$ where $(x,y)\in D$ is $A = \iint_D \sqrt{1+(f_x)^2+(f_y)^2}\,\mathrm{d}A$.

16.7 Surface integrals.

- Compute: $\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(\mathbf{r}(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}A$.
  - Note: The surface area of S is $\iint_S \mathrm{d}S = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}A$, as above.
- Special case: Integrating over $z = f(x,y)$: $\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(x,y,g(x,y))\,\sqrt{(f_x)^2+(f_y)^2+1}\,\mathrm{d}A$.
- Convention: Positive orientation is for outward normal vectors.
- Surface integrals of vector field $\mathbf{F}$, i.e. the flux of $\mathbf{F}$ across S:

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint_S (\mathbf{F}\cdot\mathbf{n})\,\mathrm{d}S = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,\mathrm{d}A.$$

- Special case: If S is the graph of $z = f(x,y)$, and $\mathbf{F} = \langle P, Q, R\rangle$, then

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint_D (-P\,f_x - Q\,f_y + R)\ \mathrm{d}A.$$

16.8 Stokes' theorem.

- Use to convert a surface integral to a line integral around the boundary, or vice-versa.
- $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_S \mathrm{curl}\,(\mathbf{F})\cdot \mathrm{d}\mathbf{S}$, for C the positively oriented boundary of S.
- Corollary: If S_1 and S_2 share the same boundary C with the same orientation, then

$$\iint_{S_1} \mathrm{curl}\,(\mathbf{F})\cdot \mathrm{d}\mathbf{S} = \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_{S_2} \mathrm{curl}\,(\mathbf{F})\cdot \mathrm{d}\mathbf{S}.$$

16.9 The divergence theorem.

- Use the divergence theorem to convert surface integrals to volume integrals, or vice-versa.
- $\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iiint_E \mathrm{div}\,(F)\ \mathrm{d}V$, for S the region bounding E with outward orientation.