Review

for 162.02 final

7.8 Improper integrals.

- Integrals can be proper, improper but convergent, or improper and divergent.
- Two types of improper integrals: discontinuous functions, infinite bounds.

10.1 Curves defined by parametric equations.

- Sketch curves, graph with calculator.
- Convert to/from cartesian coordinates.
- Find intersections of curves. They may have the same (x, y) coordinates with different parameter values.

10.2 Calculus with parametric curves.

- Derivatives: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}$.
- Arc length: $L = \displaystyle\int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$.
- Area $= \displaystyle\int_a^b y\,\mathrm{d}x$. Also area between curves (compute areas, subtract).

10.3 Polar coordinates.

- Sketch curves, graph with calculator.
- Convert to/from cartesian coordinates.
- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\,\sin(\theta) + r\,\cos(\theta)}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\,\cos(\theta) - r\,\sin(\theta)}$. Use to find tangent line, etc...

10.4 Areas and lengths in polar coordinates.

- Area $= \dfrac{1}{2}\displaystyle\int_a^b r^2\,\mathrm{d}\theta$, also area between curves (compute areas, subtract).
- Arc length: $L = \displaystyle\int_a^b \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$.

11.1 Sequences.

- Identify if a sequence is convergent, bounded, monotonic, etc...
- Monotonic sequence theorem: Every bounded monotonic sequence is convergent.
- Compute limits of convergent sequences.

11.2 Series.

- Value is the limit of the sequence of partial sums, if it exists. i.e. $\displaystyle\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty}\sum_{n=0}^{N} a_n$.
- Geometric series: $\displaystyle\sum_{n=0}^{\infty} a\,x^n = \frac{a}{1-x}$ for $|x| < 1$.
- Harmonic series: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$ diverges.

- Telescoping series: $\sum_{n=0}^{\infty}(f(n)-f(n+1)) = f(0) - \lim_{n\to\infty} f(n)$.
- Test for divergence: If $a_n \nrightarrow 0$, then $\sum_{n=0}^{\infty} a_n$ diverges. The converse is not true. (ex: Harmonic.)
- Series need not start at $n = 0$. Be comfortable working with different initial indices.

11.3 Integral test.

- Integral test: Suppose f is continuous, positive, decreasing on $[1,\infty)$, and let $a_k = f(k)$. Then $\sum_{k=1}^{\infty} a_k$ is convergent if and only if $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent.
- Remainder estimate: Suppose f and a_k are as above and $\sum a_k$ is convergent. Then the remainder $R_n = s - s_n$ satisfies $\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \le R_n \le \int_n^{\infty} f(x)\,\mathrm{d}x$.
- p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

11.4 Comparison tests.

- Comparison test: For series with positive terms, smaller than convergent converges; bigger than divergent diverges.
- Limit comparison test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $a_n/b_n \to c$, where c is a finite number bigger than 0, then the series either both converge or both diverge.

11.5 Alternating series.

- Alternating series test: If $\{a_n\}$ is a positive, decreasing sequence with limit of 0, then $\sum(-1)^n a_n$ converges.
- Alternating series estimation: The error in truncating an alternating series after n terms is no more than the absolute value of the first neglected term a_{n+1}.

11.6 Absolute convergence, root test, ratio test.

- $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ converges.
- A series that is convergent but not absolutely convergent is called conditionally convergent.
- Ratio test: Suppose $|a_{n+1}/a_n| \to L$. Note $L \ge 0$ by definition. If $L < 1$, then $\sum a_n$ converges. If $L > 1$, then $\sum a_n$ diverges. The ratio test is inconclusive if $L = 1$.
- Root test: Suppose $\sqrt[n]{|a_n|} \to L$. Note $L \ge 0$ by definition. If $L < 1$, then $\sum a_n$ converges. If $L > 1$, then $\sum a_n$ diverges. The root test is inconclusive if $L = 1$.

11.8 Power series.

- Definition.
- Use ratio test to find radius and interval of convergence. Check endpoints!

11.9 Representations of functions as power series.

- Rewrite and substitute into power series you already know.
- Integrate and differentiate power series. (Both preserve radius of convergence.)

11.10 Taylor and Maclaurin series.

- Taylor series of f centered at a is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.
- If f has a power series expression centered at a, it is necessarily its Taylor series.
- The Maclaurin series of f is the Taylor series of f centered at 0.

- Taylor's Inequality: If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies $|R_n(x)| \leq \dfrac{M}{(n+1)!}\,|x-a|^{n+1}$.
- If $R_n(x) \to 0$ for $|x-a| < R$, then f is equal to its Taylor series on $|x-a| < R$.
- Know common Maclaurin series: $\dfrac{1}{1-x}$, e^x, $\sin(x)$, $\cos(x)$, $\tan^{-1}(x)$.

11.11 Applications of Taylor polynomials.

- Use Taylor's inequality to find $T_n(x)$ to approximate a function value, estimate the error/remainder.

12.1 The third dimension.

- Distance between (x, y, z) and (a, b, c) is $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$.
- A sphere is the set of all points the same distance away from its center. That is, the equation of a sphere with center (a, b, c) is $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$, where r is the radius of the sphere.

12.2 Vectors.

- A vector $\mathbf{v}$ is a magnitude and a direction. We can write it component-wise or in terms of the basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. In 2d, $\mathbf{v} = \langle v_1, v_2 \rangle = v_1\,\mathbf{i} + v_2\,\mathbf{j}$. In 3d, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$.
- The vector from (x_1, y_1, z_1) to (x_2, y_2, z_2) is $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$. Similarly for 2d.
- The magnitude/length/norm (these are synonyms) of $\mathbf{v}$ in 3d is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. Similarly for 2d.
- A unit vector is a vector of length 1. The unit vector in the direction of $\mathbf{v} \neq \mathbf{0}$ is $\mathbf{u} = \mathbf{v}/|\mathbf{v}|$.
- Know vector arithmetic: addition, subtraction, and scalar multiplication.

12.3 Dot product.

- Dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is the scalar $\mathbf{a} \cdot \mathbf{b} = a_1\,b_1 + a_2\,b_2 + a_3\,b_3$. Similarly for 2d.
- Know properties of the dot product. In particular, note $\mathbf{a} \cdot \mathbf{a} = |a|^2$ and $\mathbf{a} \cdot \mathbf{b} = |a|\,|b|\,\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.
- $\mathbf{a}$ and $\mathbf{b}$ are orthogonal/perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.
- Projections:
  - Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is the length of the component of $\mathbf{b}$ in the $\mathbf{a}$ direction: $\text{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|a|}$.
  - Vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is the vector portion of $\mathbf{b}$ in the direction of $\mathbf{a}$: $\text{proj}_{\mathbf{a}}\mathbf{b} = (\text{comp}_{\mathbf{a}}\mathbf{b})\,\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\,\mathbf{a}$
    Note that $(\text{proj}_{\mathbf{a}}\mathbf{b}) \cdot (\mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b}) = 0$.
- Work: $W = \mathbf{F} \cdot \mathbf{D}$, where W is work, $\mathbf{F}$ is force, and $\mathbf{D}$ is displacement.

12.4 Cross product.

- Cross product is only defined for three dimensional vectors.
- $\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2\,b_3 - a_3\,b_2, a_3\,b_1 - a_1\,b_3, a_1\,b_2 - a_2\,b_1 \rangle$.
- $\mathbf{a} \times \mathbf{b}$ is a vector orthogonal to $\mathbf{a}$ and $\mathbf{b}$ in direction determined by the right-hand rule.
- $|\mathbf{a} \times \mathbf{b}| = |a|\,|b|\,\sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.
- $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.
- Know properties of cross product, especially arithmetic operations involving both cross and dot products.

- Area of parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$.
- Torque: $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ where $\boldsymbol{\tau}$ is the torque, $\mathbf{r}$ connects the pivot to the point where force is applied, and $\mathbf{F}$ is the force.

12.5 Lines and planes.

- Equation of a line. (A line is determined by two points.)
  - Vector equation: $\mathbf{r} = \mathbf{r_0} + t\,\mathbf{v}$.
  - Parametric equations: $x = x_0 + a\,t, \quad y = y_0 + b\,t, \quad z = z_0 + c\,t$.
  - Symmetric equations: $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.
  - Find equation of line passing through two points, intersections of lines, etc...
- Equation of a plane. (A plane is determined by three points, or a point and a normal vector. Planes with parallel normals are parallel.)
  - Vector equation: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$, or $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$.
  - Scalar equation: $n_1\,(x - x_0) + n_2\,(y - y_0) + n_3\,(z - z_0) = 0$.
- Find intersections of lines and planes, planes and planes, angles between them, distances from points to lines or planes, etc...

13.1 Vector functions, space curves.

- A vector function is a function that takes a scalar (e.g. a parameter t) and returns a vector.
- Find the domain of a vector function.
- The limit of a vector function is the vector of the limits.

13.2 Calculus with vector functions.

- Integrate and differentiate vector functions component-wise.
- Know properties of integration and differentiation. In particular, note that with vectors, we have two different types of products, so two product rules, both analogous to the one-dim case: $(\mathbf{u}(t) \cdot \mathbf{v}(t))' = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ and $(\mathbf{u}(t) \times \mathbf{v}(t))' = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.

13.3 Arc length and curvature.

- The length of the curve defined by $\mathbf{r}(t)$, $a \le t \le b$ is $L = \displaystyle\int_a^b |\mathbf{r}'(t)|\,\mathrm{d}t$.
- Reparameterize a curve with respect to arc length.
- The curvature of the curve $\mathbf{r}(\tau)$ at time t is $\kappa(t) = \left|\dfrac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right| = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$, where $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ is the unit tangent vector. The curvature of $y = f(x)$ at x is $\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$.
- Normal vector to $\mathbf{r}(t)$ is $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. Binormal vector to $\mathbf{r}(t)$ is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$. Normal and binormal vectors at a point determine the normal plane at that point.

13.4 Velocity and acceleration.

- If a particle's position is $\mathbf{r}(t)$, then its velocity function is $\mathbf{v}(t) = \mathbf{r}'(t)$, and its acceleration is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.
- Newton's second law: The acceleration $\mathbf{a}$ resulting from force $\mathbf{F}$ acting on an object of mass m satisfies $\mathbf{F} = m\,\mathbf{a}$.
- Decomposing acceleration into tangential and normal directions: $\mathbf{a} = v'\,\mathbf{T} + \kappa\,v^2\,\mathbf{N}$, where $v = |\mathbf{v}|$.