Topics

for 161.01 final

2.1 The tangent and velocity problems.

- Estimating limits from tables.
- Instantaneous velocity is limit of average velocity.
- Slope of tangent line is limit of slope of secant lines.

2.2 The limit of a function.

- Intuitive definition: limit is the y value approached as the x values go towards a point.
- Be careful guessing limits numerically: round off errors or simply picking points too far away may cause trouble.
- Left-hand, right-hand, and two-sided limits. (If left and right limits are different, two sided limit does not exist.)
- Undamped oscillations at arbitrarily high frequencies do not have limits (e.g. $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ does not exist.)
- If function values get arbitrarily large as x value is approached, limit is said to be $+\infty$. Similarly for $-\infty$.
- Identify vertical asymptotes based on infinite limits. (Vertical asymptotes are lines, not numbers.)

2.3 Calculating limits using the limit laws.

- Limits of sum, difference, or product is respectively the sum, difference, or product of the limits provided they both exist.
- Limit of a constant times a function is the constant times the limit of a function (special case of product.)
- Limit of a quotient is the quotient of the limits provided they both exist and the lower limit is not 0.
- Squeeze theorem: If $f \leq g \leq h$, $f \to L$ and $h \to L$ as $x \to a$, then $g \to L$ as $x \to a$.

2.4 The precise definition of a limit.

- $\lim_{x\to a} f(x) = L$ means for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Similar definitions hold for left and right limits.
- Be able to use the definition to prove the limit for a linear problem, e.g. $\lim_{x\to 2}(4x - 1)$.

2.5 Continuity.

- A function f is continuous at a if $f(a)$, the limit as $x \to a^-$ and the limit as $x \to a^+$ all exist and are equal.
- We can exchange the order of limits and continuous functions.
- Polynomials, trig functions, rational functions, root functions, exponential functions, and log functions are continuous on their domains.
- Composition of continuous functions is continuous.
- Intermediate value theorem.
  - Intuitively: A continuous function cannot go from one value to another without going through the values in between.
  - To show two functions are equal on an interval, show their difference must be 0 somewhere.

2.6 Limits at infinity; horizontal asymptotes.

- Limits at ∞ (or $-\infty$) describe how a function behaves for large (or small, respectively) x-values.
- Horizontal asymptotes are the lines that a function approaches as $x \to \pm\infty$. (At most two horizontal asymptotes.)
- Do algebra to resolve indeterminate form problems ($\infty - \infty$, $0/0$, ∞/∞, etc ...).
- Be able to find tangent lines and normal lines.

2.7 Derivatives and rates of change.

- The derivative $f'(a)$ is defined: $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.
- Be able to find derivatives using the limit definition, especially of polynomials, radicals, and rational functions.
- The derivative indicates the instantaneous rate of change. Thus the derivative of position is velocity, the derivative of velocity is acceleration, the derivative of acceleration is jerk, the derivative of a function is the slope of its tangent line, etc...

2.8 The derivative as a function.

- Be comfortable with both Newton's and Leibnitz's notations.
- $f'(x) = \lim_{x_2 \to x} \dfrac{f(x_2) - f(x)}{x_2 - x} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.
- Not all functions are differentiable at every point. (e.g. $f(x) = |x|$ is not differentiable at 0.)
- If f is differentiable at a, then f is continuous at a.
- The second derivative is the derivative of the derivative, the third derivative is the derivative of the second derivative, etc...
- Be able to estimate the value of a derivative from a graph or table.

3.1 Derivatives of polynomials and exponential functions.

- Power rule, constant rule, constant multiple rule, sum and difference rules.
- The derivative of e^x is e^x. (Except for constant multiples of e^x, no other function is its own derivative.)

3.2 The product and quotient rules.

- $(fg)' = f'g + fg'$, $\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$.
- Sometimes it is easier to algebraically simplify an expression and then differentiate instead of using the quotient rule.

3.3 Derivatives of trigonometric functions.

- $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$, $\lim_{x \to 0} \dfrac{1 - \cos x}{x} = 0$, and more complicated combinations.
- Derivatives of trig functions: sin, cos, tan, csc, sec, cot.

3.4 The chain rule.

- Suppose $F(x) = f(g(x))$. Then $F'(x) = f'(g(x))g'(x)$.
- Suppose $y = f(u)$ where $u = g(x)$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.
- $\dfrac{d}{dx}(a^x) = a^x \ln a$.

3.5 Implicit differentiation.

- Use implicit differentiation to find first and second derivatives.
- Derivatives of inverse trig functions: $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, $\csc^{-1}$, $\sec^{-1}$, $\cot^{-1}$.

3.6 Derivatives of logarithmic functions.

- $(\ln x)' = 1/x$.
- Since $\log_b x = (\ln x)/(\ln b)$, it follows that $(\log_b x)' = 1/(x \ln b)$.
- Logarithmic differentiation: Take logs of both sides, implicitly differentiate, then solve for y'. Good for complicated products, quotients, and powers.

3.7 Rates of change in the natural and social sciences.

- Derivative is the rate of change. Biology, physics, chemistry, etc... all study things that change.
- Position/velocity/acceleration/jerk problems, such as a tossed ball.
- Marginal revenue/cost problems.
- Density problems.
- Be able to adapt to other problem types.

3.8 Exponential growth and decay.

- If $y' = ky$, then $y(t) = y(0)e^{kt}$.
- Many processes can be modeled with this relation, including: radioactive decay, population growth, temperature changes, and compound interest.

3.9 Related rates.

- Write general formulas, take the derivative (use the chain rule), then plug in your data and solve for the missing pieces.
- Common tricks include: similar triangles, law of cosines.
- Best to always work in radians (because the derivatives of sin and cos are different if using degrees).

3.10 Linear approximations and differentials.

- Near the point of tangency, the tangent line approximates the graph of the function. The formula for the tangent line is called the linearization of the function at the point.
- If $y = f(x)$, then the differential is $dy = f'(x)dx$.
- Use differentials to estimate error; use linearization or differentials to estimate function values.

4.1 Maximum and minimum values.

- Critical points are places in the domain where the derivative is zero or undefined.
- Local extrema can only happen at critical points. (A critical point need not be an extrema.)
- Absolute extrema happen at end points or at critical points.

4.2 The mean value theorem.

- Rolle's theorem: If f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then there is a c in (a, b) such that $f'(c) = 0$.
  - i.e. If two points have the same y value, then somewhere in between the function has a horizontal tangent. (Under certain basic assumptions.)
- Mean value theorem: If f is continuous on $[a, b]$ and differentiable on (a, b) then there is a c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

- i.e. Somewhere the slope of a (continuous and differentiable) function is the same as the slope of the secant line.
- Use MVT to show inequalities, prove the number of roots a function has.
- Rolle's theorem is a special case of MVT.

- If $f' = g'$, then $f(x) = g(x) + C$ for some constant C.

4.3 How derivatives affect the shape of a graph.

- If $f' > 0$ then f is increasing; if $f' < 0$, then f is decreasing.
- First derivative test: if $f' > 0$ to the left of a critical point and $f' < 0$ to the right, then we switched from increasing to decreasing, so the critical point is a local max. If instead f' switches from negative to positive, then the critical point is a minimum.
- If $f'' > 0$, then f is concave up (i.e. the graph of f lies above all its tangents.) If $f'' < 0$, then f is concave down.
- Inflection points are points where the concavity changes.
- Second derivative test: Suppose c is a critical point. If $f''(c) > 0$, then a local min at $x = c$. If $f''(c) < 0$, then a local max at $x = c$. If $f''(c) = 0$, then the second derivative test provides no information; try the first derivative test.

4.4 L'Hospital's rule.

- If $0/0$ or ∞/∞ in limit, take the derivative of the top and the bottom, then take the limit, otherwise just plug in if quotient.
- Turn indefinite multiplication forms into division forms by dividing by the reciprocal instead of multiplying.
- Turn addition or subtraction into multiplication or division by getting a common denominator or other algebraic manipulation.
- Turn exponential into multiplication by taking the natural log, evaluating the limit, then taking the exponential of the result.

4.5 Summary of curve sketching.

- Look for: domain, intercepts, symmetry, asymptotes (you may need to use L'Hospital's rule), intervals of increasing and decreasing, local max and mins, and concavity and points of inflection.

4.6 Graphing with calculus and calculators.

- Use calculus to pick a suitable window (or windows) to graph in; you want to show all the features listed in 4.5 above.

4.7 Optimization problems.

- Word problems.
- Strategy: Draw a diagram, introduce variables for distances etc in your diagram, find algebraic relationships between the variables, express desired quantity in terms of one variable, check critical points and end points for maxes and mins.

4.9 Antiderivatives.

- F is an antiderivative (or indefinite integral) of f if $F' = f$. (i.e. an antiderivative of f is a function whose derivative is f.)
- When asked for a general antiderivative of a function, find a particular one and add an arbitrary constant C.

5.1 Areas and distances.

- Approximate area using a finite number of rectangles, left-, right- and mid-points.
- Area under the graph is limit of the sum of the areas of approximating rectangles.
- Σ-notation.
- Distance is area under velocity function.

5.2 The definite integral.

- Definition of a definite integral. (Riemann sums: know the difference between left-, right-, and midpoint sums).
- Σ-notation formulas for $\sum 1$, $\sum i$, $\sum i^2$, $\sum i^3$.
- Properties of the integral.

5.3 The fundamental theorem of calculus.

- Derivative of $\int_a^x f(t)\,dt$, and more complicated problems involving a chain rule part.
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$, where F is any antiderivative of f.

5.4 Indefinite integrals and the net change theorem.

- Indefinite integrals are the same as antiderivatives, and unique up to $+C$.
- Don't forget to write the $+C$.
- Integral of a rate of change is the net change: $\int_a^b F'(t)\,dt = F(b) - F(a)$, this is a rephrasing of the second fundamental theorem.
- Verify antiderivative formulas by differentiating.

5.5 The substitution rule.

- u-substitution is the reverse of the chain rule: If $u = g(x)$, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

- For indefinite integrals, change back to the original variable.
- Integrals of even functions, of odd functions (might be hidden).

6.1 Areas between curves.

- The area of curves that don't cross is the integral of (top-bottom) or (right-left).
- Break into multiple pieces if the curves cross.

6.2 Volumes.

- Discs (integrate πr^2), washers (integrate $\pi R_{\text{big}}^2 - \pi R_{\text{small}}^2$), and other cross-sections (integrate $A(x)$, the cross-sectional area).

6.3 Volumes by cylindrical shells.

- Integrate $2\pi rh$.
  - ex: $y = f(x)$ from a to b rotated about $x = c$ has volume $V = 2\pi \displaystyle\int_a^b (x - c)f(x)\,dx$.

6.4 Work.

- Work is force times distance, although for our purposes at least one of these is going to vary, so you'll have to integrate.
- Hooke's law (spring problems): $f = kx$. Note x is the distance the spring is stretched beyond its natural length.
- Common problems: springs, pumping water, lifting chain.
- Remember: pounds is a unit of force, kilograms is a unit of mass. When working with metric problems, multiply mass by $g \approx 9.8$ to get a force, but do not multiply pounds by gravity; they already include it.

6.5 Average values.

- The average of f on $[a, b]$ is $\bar{f} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.
- Mean value theorem for integrals: If f is continuous on $[a, b]$, then there is a c in $[a, b]$ such that $f(c) = \bar{f}$. (Somewhere the function equals its average value.)

7.1 Integration by parts.

- This is the reverse process of the product rule.
- $\displaystyle\int u\,dv = uv - \int v\,du$, or for definite integrals, $\displaystyle\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$.
- You may need to use this process more than once or in combination with other techniques.
- Strategy: Try LIATE for picking u. That is, take u to be whichever of the following comes first: (L)ogs, (I)nverse trig, (A)legebraic functions, (T)rig functions, or (E)xponential functions.
- If your choice of u and dv does not work, try a different one.
- Tricky problems, such as $\int e^x \sin(x)dx$.

7.2 Trig integrals.

- When integrating $\sin^m(x)\cos^n(x)dx$, if m is odd, turn all but one sine into a cosine via $\sin^2(x) = 1 - \cos^2(x)$, let $u = \cos(x)$. Similarly if n is odd. If both m and n are even, use half-angle formulas: $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$, and $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$.
- For $\int \tan^m(x)\sec^n(x)\,dx$, if n is even, keep a $\sec^2$, turn the rest into tangents via $\sec^2(x) = 1 + \tan^2(x)$, let $u = \tan(x)$. If m is odd, keep a $\sec(x)\tan(x)$, turn the remaining tangents into secants via $\tan^2(x) = \sec^2(x) - 1$, let $u = \sec(x)$. Other cases are harder.
- For $\int \sin(mx)\cos(nx)dx$, and similar problems, eliminate the product via one of:
  (a) $\sin(A)\cos(B) = \frac{1}{2}(\sin(A-B) + \sin(A+B))$
  (b) $\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$
  (c) $\cos(A)\cos(B) = \frac{1}{2}(\cos(A-B) + \cos(A+B))$

7.3 Trig substitution.

- Useful for problems involving square roots. You may need to complete the square on the inside.
- For $\sqrt{a^2 - x^2}$, let $x = a\sin(\theta)$.
- For $\sqrt{a^2 + x^2}$, let $x = a\tan(\theta)$.
- For $\sqrt{x^2 - a^2}$, let $x = a\sec(\theta)$.
- Use right-triangle trig to convert antiderivative back to original variables.

7.4 Partial fractions.

- If degree of numerator is bigger than or equal to degree of denominator, do long division first.
- Factor denominator completely. For linear terms, use a numerator of form A; for irreducible quadratics use a numerator of the form $A\,x + B$.
- If a factor of the denominator is raised to some power, use multiple terms with all integer exponents between 1 and that power.
- Sometimes you need to make u-substitutions before you can do partial fractions.
  - If integrand involves $\sqrt[n]{g(x)}$, try letting $u = \sqrt[n]{g(x)}$. (Rationalizing substitution.)

7.5 Strategy for integration.

- Look for algebraic simplifications or manipulations at each step.
- Primary techniques are u-substitution and integration by parts. Use these to reduce the problem to an easier integral, repeat. Look out for opportunities to use partial fractions, trig integrals, or trig substitution.

8.1 Arc length.

- $L = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x = \int_c^d \sqrt{1 + (g'(y))^2}\,\mathrm{d}y$.
- Arc length function (arc length from a point): $s(x) = \displaystyle\int_a^x \sqrt{1 + (f'(t))^2}\,dt$.

8.2 Surface area of solids of revolution.

- When rotating $f(x)$ around x-axis, use $S = 2\pi \int_a^b f(x)\,\sqrt{1 + (f'(x))^2}\,\mathrm{d}x$.
- When rotating $g(y)$ around y-axis, use $S = 2\pi \int_c^d g(y)\,\sqrt{1 + (g'(y))^2}\,\mathrm{d}y$.
- You may need to solve for x or y.