Dynamics 101

We consider autonomous (that is, not t-dependent) one and two dimensional differential equations. i.e. Equations of the forms

$$\dot{x} = f(x), \quad \text{or} \tag{1}$$

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y), \end{aligned} \tag{2}$$

where the dot denotes differentiation with respect to t. Let us assume that f and g are continuous.

The 1-D case

Note that in (1), if at one moment in time $x = \overline{x}$ where $f(\overline{x}) = 0$, then $\dot{x} = 0$, that is, x is not changing with respect to time. Hence an instant later, we still have $x = \overline{x}$. Such an $\overline{x}$ is called a **fixed point** of the equation, because solutions get "stuck" there. (For linear autonomous systems, once a fixed point is reached, the system will remain there for all time t. The same does not necessarily hold for nonlinear systems; consider $\dot{x} = x^{1/3}$, which has an infinite number of solutions through the point $(t, x) = (0, 0)$.)

A fixed point $\overline{x}$ is said to be **asymptotically stable** if solutions that start near $x = \overline{x}$ approach it as time increases, and **unstable** if they move away from it.

Theorem: Suppose $\overline{x}$ is a fixed point of (1). If $f'(\overline{x}) < 0$ then $\overline{x}$ is an asymptotically stable fixed point. If $f'(\overline{x}) > 0$ then $\overline{x}$ is unstable.

Note that one-dimensional autonomous systems cannot oscillate because that would require x to change from increasing ($\dot{x} > 0$) to decreasing ($\dot{x} < 0$). At some point in between, we would have $\dot{x} = 0$ and hence be stuck at a fixed point. Thus the only types of solutions for (1) are constant solutions (stuck at a fixed point), solutions that move monotonically toward (and may or may not reach) a fixed point, and solutions that tend toward $\pm\infty$.

The 2-D case

In (2), the set of points (x, y) where $\dot{x} = 0$ (i.e. where $f(x, y) = 0$) is called the **x-nullcline**. Similarly, the set of points (x, y) where $\dot{y} = 0$ (i.e. where $g(x, y) = 0$) is called the **y-nullcline**. Note that solutions can only cross the x-nullcline vertically, and the y-nullcline horizontally.

The point $(\overline{x}, \overline{y})$ is a **fixed point** of (2) if $\dot{x}|_{(\overline{x},\overline{y})} = \dot{y}|_{(\overline{x},\overline{y})} = 0$, i.e. if $f(\overline{x}, \overline{y}) = g(\overline{x}, \overline{y}) = 0$. Note that fixed points correspond to the intersections of the nullclines.

Theorem: Suppose $(\overline{x}, \overline{y})$ is a fixed point of (2). Let A denote the Jacobian of (2) evaluated at the fixed point, i.e.

$$A = \left.\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}\right|_{(x,y)=(\overline{x},\overline{y})},$$

where f_x denotes the partial derivative of f with respect to x. Let λ_1 and λ_2 be the eigenvalues of A, i.e. the roots of $\det(\lambda I_2 - A) = 0$. If $\mathrm{Re}(\lambda_1) < 0$ and $\mathrm{Re}(\lambda_2) < 0$, then $(\overline{x}, \overline{y})$ is asymptotically stable. If $\mathrm{Re}(\lambda_1) > 0$ and $\mathrm{Re}(\lambda_2) > 0$, then $(\overline{x}, \overline{y})$ is unstable. If one of the eigenvalues has positive real part and the other has negative real part, then the fixed point is a **saddle point**, that is, trajectories approach it in some directions and are repelled from it in other directions.

In two dimensions, we can have all the types of solutions that were possible in the one-dimensional case, and we can also have periodic solutions. (Adding a third variable will allow for chaos.)

Warning: All of the above generalizes easily for higher dimensional systems; the following theorems do not.

Poincaré-Bendixson Theorem:
Let P be a closed, bounded subset of $\mathbb{R}^2$, not containing any fixed points of (2). Suppose further that there exists a trajectory confined in P. Then any trajectory confined in P is either a periodic orbit or approaches a periodic orbit. In particular, there exists a periodic orbit.

Dulac's Criterion:
Let D be a simply connected subset of $\mathbb{R}^2$. Let $B(x, y)$ be a function defined on D. If $(Bf)_x + (Bg)_y$ is either always strictly positive or always strictly negative, then there are no periodic orbits lying in D.